Topics in Analysis: Optimal Transport	**Semester II 2024/25**	**Time: 3 hours**
(Open Notes in ones own handwriting	**Final Exam**	**Date: 22nd April, 2024.**

1. Let $d = 1$ and consider the probability measures on $\mathbb{R}$ given by

$$\mu \sim \text{Uniform}[0,1], \quad \nu_n = \left(1 - \frac{1}{n}\right)\text{Uniform}[0,1] + \frac{1}{n}\delta_n(\cdot),$$

where δ_n is the dirac measure at n.

   (a) (5 points) Show that $\nu_n \xrightarrow{w} \mu$ as $n \to \infty$.

   (b) (10 points) Find T_n the optimal transport map from μ to ν_n

   (c) (5 points) Show that $W_2(\mu, \nu_n) \geq n$ and conclude that $\mathcal{W}_2(\mu, \nu_n) \not\to \mathcal{W}_2(\mu, \nu)$ as $n \to \infty$.

2. Let $d \in \mathbb{N}$ and $\mathcal{P}_2(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ which have second moment. For each of the following find the optimal transport map from μ to ν.

   (a) (10 points) $\mu = \text{Uniform}([0,1]^d), \nu = \text{Uniform}([0,r]^d)$ for some $r \in \mathbb{R}_{>0}$.

   (b) (10 points) $\mu = \mathcal{N}(0, I), \nu = \mathcal{N}(a, I)$ in $\mathbb{R}^d$ some $a \in \mathbb{R}$.

3. Let $\{\theta_t\}_{t\geq 0}$ be a continuous curve in $\mathbb{R}^d$.

   (a) (10 points) Show that the curve $t \mapsto \mu_t := \delta_{\theta_t}(\cdot)$ is continuous on $(\mathcal{P}(\mathbb{R}^d), \mathcal{W}_2(\mathbb{R}^d))$.

   (b) (10 points) Can you impose additional conditions of θ_t so that the curve in (a) is absolutely continuous ?

4. Let $\mathcal{P}_2(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ which have second moment. Consider the lower semi continuous function , $\mathscr{F} : \mathcal{P}_2(\mathbb{R}^d) \to [0, \infty)$, given by

$$\mathscr{F}(\mu) = \frac{1}{2}\int_{\mathbb{R}^d} ||x||^2 d\mu(x)$$

   (a) (5 points) Find the minimizer of $\mathscr{F}$

   (b) (5 points) Find

$$\nabla_{\mathcal{W}}(\mathscr{F})(\mu)(x), \forall\, x \in \mathbb{R}^d$$

   and the continuity equation for the gradient flow.

   (c) (5 points) Show that

$$X_t^x = \exp(-t)x$$

   is the unique solution to the Cauchy problem ,

$$\dot{X_t}^x = -(X_t^x)\ ,\ X_0^x = x$$

(d) (5 points) Assume $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ and $X_0 \sim \mu_0$. Now define, $\mu_t = X_{t\#}\mu_0$ (Push forward of μ_0 via the map X_t). Show that $\mathcal{W}_2(\mu_t, \delta_0) \leq e^{-t}\mathcal{W}_2(\mu_0, \delta_0)$ for all $t > 0$ where $\mathcal{W}_2$ is the Wasserstein-2 metric.

5. Let $\mathcal{P}_2(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ which have second moment. Consider the function, $\mathscr{F} : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$, given by

$$\mathscr{F}(\mu) = \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} ||x||^4 d\mu(x) & \text{if finite.} \\ \\ \infty & \text{otherwise} \end{cases}$$

(a) (5 points) Find the minimizer of $\mathscr{F}$

(b) (5 points) Find $\nabla_{\mathcal{W}}(\mathscr{F})(\mu)(x)$ and identify the continuity equation for the gradient flow w.r.t. $\mathscr{F}$.

(c) (5 points) Show that

$$X_t^x = \frac{1}{\sqrt{2t + 1/x^2}}$$

is the unique solution to the Cauchy problem ,

$$\dot{X}_t^x = -(X_t^x)^3 \ , \ X_0^x = x$$

(d) (5 points) Assume μ_0 is in the domain of $\mathscr{F}$ and $X_0 \sim \mu_0$. Find the law of $\mu_t = X_{t\#}\mu_0$ in terms of X_0 and conclude that the convergence to the minimizer is not an exponential rate in the Wasserstein-2 metric.